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Relation of The New Calogero Models and xxz Spin Chains

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February 7, 2008

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Abstract

We extend our previous analysis of the classical integrable models of Calogero in several respects. Firstly we provide the algebraic resaons of their quantum integrability.Secondly we show why these systems allow their initial value problem to be solved in closed form . Furthermore we show that due to their similarity with the above models the classical and quantum Heisenberg magnets with long rang interactions in a magnetic field are also intergrable. Explicit expressions are given for the integrals of motion in involution in the classical case and for the commuting operators in the quantum case.

1 Introduction

Motivated by the discovery of Camassa, Holm, and Hyman [1] that the classical system characterized by the Hamiltonian

$$H = \sum_{j,k=1}^{N} p_j p_k exp[-\eta|q_j - q_k|] \tag{1}$$

with the usual poisson brackets is integrable, Calogero investigated the the problem of integrabiltiy of the more general Hamiltonians

$$H = \sum_{j,k=1}^{N} p_j p_k f(q_j - q_k) \tag{2}$$

with conserved quantities of the form

$$c_{jk} = p_j p_k g(q_j - q_k) \tag{3}$$

and succeeded to find a class of such systems [2], where

$$f(x) = \lambda + \mu \cos(x) \tag{4}$$

and

$$g(x) = 1 - \cos(x). \tag{5}$$

(Later he also investigated the classical [3] and quantum [4] solvability integrability of other similar systems where the factor $p_j p_k$ appears under a square root.) It was shown in [5] that the integrability of systems of type (2,4) is due to a very simple algebraic structure , i.e: the existence of N copies of $su(1,1)$ algebras one embeded inside the other, such that the Casimir operators of all copies commute with each other. These Casimirs were shown to be the integrals

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of motion of the system (2) which are in involution with each other. The purpose of this paper which is a continuation of [5] is threefold.

1) I extend my previous considerations to the quantum case and show that quantum versions of these systems and of more general systems are also integrable .

2) As a model with physical interest I study by algebraic methods the integrability of classical and quantum Heisenberg magnets with long rang interactions in a magnetic field .

3) It is known that integrability and " solvability in closed form" do not always imply each other at least in practice. The models introduced in [2] have the merit that they are integrable (i.e: possess N independent integrals of motion in involuton) and at the same time sovable (i.e: admit their initial value problem to be solved in closed form). Hence there must be a simple algebraic reason why these models are also solvable in the above sence . We will provide the rational behind their solvability.

2   Classical xxz Heisenberg Magnet

The strategy that we follow is to take a system of classical vectors interacting with each other and with an external field. The dynamical variables are given by a vector of length squared equal to a constant C_1 .

$$\mathbf{S}_i = (S_i^a; a = 1, 2, 3 \quad \mathbf{S}_i \cdot \mathbf{S}_i = C_1) \quad i = 1, 2, ...N \tag{6}$$

subject to the poisson bracket relations:

$$\{S_i^a, S_j^b\} = \epsilon^{abc} S_j^c \delta_{ij} \tag{7}$$

The Hamiltonian is given by :

$$H = \sum_{j,k=1}^{N} \lambda S_j^z S_k^z + \mu(S_j^x S_k^x + S_j^y S_k^y) + B \sum_{i=1}^{N} S_i^z \tag{8}$$

Mathematically the poisson bracket relations (7) are related with the Lie algebra $su(2)$. More presizely [6] the Lie algebra $su(2)$ induces (7) as a natural poisson bracket on it's dual which can be thought of as a Cartesian three dimensional space with local coordiantes (S^1, S^2, S^3). However this poisson bracket is degenerate , since there are functions which poisson commute with everything , g.e:

$$\{\mathbf{S.S}, S^a\} = 0$$

To obtain non-degenerate poisson bracket one should restrict oneself to those submanifolds on which $S \cdot S$ aquires a constant value C_1 i.e: the symplectic leaves.

This system is general enough for our considerations. It will have several subcases of interest:

case a) S_i^x, S_i^y, and S_i^z are real with C_1 equal to a real number say one. The symplectic leaves are two dimensional spheres. This is the classical Heisenberg Magnet describing a system of classical spins interacting whit each other and with a magnetic field B in the z direction. For $\lambda = \mu$ we have the isotropic magnet with $su(2)$ symmetry.

case b) S_i^x and S_i^y real and S_i^z pure imaginary with $C_1 = 0$. When $B = 0$, this is the system introduced by Calogero in [2] which is related to the double cone symplectic leaf

$$S^{x2} + S^{y2} + S^{z2} = 0$$

of the lie algebra $su(1,1)$. The canonical coordinates on this leaf are

$$S^z = ip \quad S^x = p\cos q \quad S^y = p\sin q \tag{9}$$

and the Hamiltonian is given by (2,3) i.e:

$$H = \sum_{j,k=1}^{N} \lambda p_j p_k + \mu \{ p_j p_k \ \cos[(q_j - q_k)] \} \tag{10}$$

with the integrals of motion given by :

$$h_m = \sum_{j,k=1}^{m} p_j p_k (1 - \cos[(q_j - q_k)]$$

and

$$P = \sum_{i=1}^{N} p_i$$

case b) S_i^x and S_i^y real and S_i^z pure imaginary with $C_1 \neq 0$, . This is the system introduced

in [5] which is related to the hyperboloidal symplectic leaf

$$S^{x2} + S^{y2} + S^{z2} = C_1$$

of the lie algebra $su(1,1)$. The canonical coordinates on this leaf are

$$S^z = ip \quad S^x = \sqrt{p^2 + C_1} \cos q \quad S^y = \sqrt{p^2 + C_1} \sin q \tag{11}$$

and the Hamiltonian is given by :

$$H = \sum_{j,k=1}^{N} \lambda p_j p_k + \mu \{ \sqrt{p_j^2 + C_1} \ \sqrt{p_k^2 + C_1} \ \cos[(q_j - q_k)] \} + B \sum_{j=1}^{j=N} p_j \tag{12}$$

with the integrals of motion given by :

$$h_m = \sum_{j,k=1}^{m} p_j p_k - \{ \sqrt{p_j^2 + C_1} \ \sqrt{p_k^2 + C_1} \ \cos[(q_j - q_k)] \}$$

and

$$P = \sum_{i=1}^{N} p_i$$

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All the results that we will obtain will apply to the above systems after minor redefinitions of constants.

In order to understand in a systematic manner the integrable structure of this system , we proceed as follows and define the variables:

$$X_1^a = S_1^a$$

....

$$X_m^a = S_1^a + S_2^a + ...S_m^a \tag{13}$$

......

$$X_N^a = S_1^a + S_2^a + ...S_N^a$$

where $a \in (1 \equiv x, 2 \equiv y, 3 \equiv z)$

It is obvious that for each m these sets of variables satisfy the same relations among themselves as in (7) and form a copy of $su(2)$ algebra , and furthermore since the smaller copies of the algebra are embeded in the larger copies we have:

$$\{X_m^a, X_n^b\} = \epsilon^{abc} X_{(m,n)}^c \tag{14}$$

where (m, n) is meant to denote the minimum of m and n

Defining for each copy, say the m-th one the Casimir function

$$C_m = \sum_{a=1}^3 X_m^a X_m^a \tag{15}$$

we obtain:

$$\{C_i, X_j^b\} = 2\epsilon^{abc} X_i^a X_{(i,j)}^c \tag{16}$$

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$$\{C_i, C_j\} = 4\epsilon^{abc} X_i^a X_j^b X_{(i,j)}^c \tag{17}$$

We now note that in the last formula the indices i and j are not dummy variables ,however the index (i,j) is either equal to i or to j , in any case the tensor which is contracted with ϵ^{abc} is symmetric with respect to the interchange of two of the indices (a,c) or (b,c) , hence the right hand side identically vanishes:

$$\{C_i, C_j\} = 0$$

It is interesting to note that although the Casimir of one copy does not commute with the generators of another copy as seen from (16) ,the Casimirs of different copies commute among themselves. However it should be noted from (16) that all the Casimir functions commute with the generators of the largest copy. i.e:

$$\{C_i, X_N^a\} = 0 \tag{18}$$

The Hamiltonain can now be written as:

$$H = \lambda Z_N^2 + \mu(X_N^2 + Y_N^2) + B Z_N \tag{19}$$

or

$$H = (\lambda - \mu) Z_N^2 + B Z_N + \mu C_N \tag{20}$$

It is seen that there are N integrals of motion in this system which are in involution with each other and with the Hamiltonian. These are

$$I = \{C_2, C_3, ...C_N, Z_N\} \tag{21}$$

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We have found enough integrals to claim integrability of the system. The explicit expressions of C_m are

$$C_m = \sum_{j,k=1}^{m} S_j^x S_k^x + S_j^y S_k^y + S_j^z S_k^z \tag{22}$$

and Z_N is the total spin (Magnetization) in the z-direction.

3   The Inintial Value Problem

In terms of the new variables one can easily write the equations of motions. From (16) wee have :

$$\{C_N, X_j^b\} = 2\epsilon^{abc} X_N^a X_j^c \tag{23}$$

or

$$\{C_N, X_j\} = 2(Z_N Y_j - Y_N Z_j) \tag{24}$$

$$\{C_N, Y_j\} = 2(X_N Z_j - Z_N X_j) \tag{25}$$

$$\{C_N, X_j\} = 2(Y_N X_j - X_N Y_j) \tag{26}$$

From which we can obtain the equations of motion .

$$\begin{pmatrix} \frac{d}{dt}X_j \\ \frac{d}{dt}Y_j \\ \frac{d}{dt}Z_j \end{pmatrix} = \begin{pmatrix} 0 & -B - 2\lambda Z_N & 2\mu Y_N \\ B + 2\lambda Z_N & 0 & -2\mu Y_N \\ -2\mu Y_N & 2\mu X_N & 0 \end{pmatrix} \begin{pmatrix} X_j \\ Y_j \\ Z_j \end{pmatrix} \tag{27}$$

which is of the form

$$\frac{d}{dt}\Psi_j = A\Psi_j \tag{28}$$

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The time dependence of the matrix A is itself determined from :

$$\frac{d}{dt}X_N = \{X_N, H\} = -(2(\lambda - \mu)Z_N + B)Y_N \tag{29}$$

$$\frac{d}{dt}Y_N = \{Y_N, H\} = (2(\lambda - \mu)Z_N + B)X_N \tag{30}$$

$$\frac{d}{dt}Z_N = \{Z_N, H\} = 0 \tag{31}$$

In any case Z_N is constant and the dynamics of X_N and Y_N depend on the time dependence of the magnetic field. For a constant magnetic field the components X_N and Y_N have a simple time evolution:

$$X_N = A\cos(\omega t + \alpha) \quad Y_N = A\sin(\omega t + \alpha) \tag{32}$$

where $\omega = 2(\lambda - \mu)Z_N + B$ and A and α are determined from the initial conditions. Once the time dependence of A is determined, the time dependence of Ψ_j will be determined from (28):

$$\Psi_j(t) = Texp\int_0^t A(t')dt'\Psi_j(0) \tag{33}$$

However one can go beyond this formal solution. We note that A can be written as follows:

$$A = -\omega L_3 - 2\mu(X_N L_1 + Y_N L_2) \tag{34}$$

where the matrices $J_1 = iL_1$ $J_2 = iL_2$ and $J_3 = iL_3$ are the generators of rotation. Inserting (32) in (34) we find that :

$$A(t) = i\omega J_3 + i\mu A(e^{i(\omega t+\alpha)}J_- + e^{-i(\omega t+\alpha)}J_+) \tag{35}$$

where $J_\pm = J_1 \pm iJ_2$. The dynamics of A is just a simple rotation around the z axis:

$$A = e^{-i(\omega t+\alpha)J_3}A(0)e^{i(\omega t+\alpha)J_3} \equiv U^{-1}(t)A(0)U(t) \tag{36}$$

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Multiplying both sides of this equation by $U(t)$ and defining $\Phi_i(t) = U(t)\Psi_i(t)$ we find:

$$\frac{d}{dt}\Phi_i(t) = (A(0) - \frac{d}{dt}U(t)U(t))\Phi_i(t)$$

$$= (A(0) - i\omega J_3)\Phi_i(t) \tag{37}$$

This is a simple evolution equation governed by a constant matrix the solution of which is simply

$$\Phi_i(t) = e^{(A(0)-i\omega J_3)t}\Phi_i(t)$$

We have thus arrived at a closed form solution of the initial value problem for Φ_i and thus for Ψ_i. Our analysis shows clearly why the initial value problem in [2] can be solved in closed form.

4   The Quantum Case

In the quantum case the field variables S_i^a are replaced by spin operators acting on a Hilbert space $V = V^{\otimes N}$ where V is an irreducible representation space of $su(2)$.(we do not restrict ourselves to the spin $\frac{1}{2}$ representation and :

$$s_i^a = 1 \otimes 1 \otimes ....1 \otimes s^a \otimes .... \otimes 1$$

Where s^a acts on the i-th space . The poisson brackets are replaced by

$$[S_i^a, S_j^b] = i\epsilon^{abc}S_j^c\delta_{ij} \tag{38}$$

and the hamiltonian has the same form as before. The role of symplectic leaves $\mathbf{S}.\mathbf{S} = C_1$ is now played by a particular irreducible representation where the Casimir operator takes a constant

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value. Again for discussion of integrability we define operators

$$X_m^a = s_1^a + s_2^a + ...s_m^a \tag{39}$$

Exactly as in the classical case we obtain

$$[X_m^a, X_n^b] = i\epsilon^{abc} X_{(m,n)}^c \tag{40}$$

Note that each operator X_m^a say Z_m acts like the third component of total spin operator in the first m spaces and acts trivially in the rest of spaces. Defining now the operators

$$C_m = X_m^a X_m^a$$

(where a sum over the index a is understood) we obtain

$$[C_m, X_n^b] = i\epsilon^{abc}(X_m^a X_{(m,n)}^c + X_{(m,n)}^c X_m^a) \tag{41}$$

$$[C_i, C_j] = i\epsilon^{abc}\left(X_n^b(X_m^a X_{(n,m)}^c + X_{(n,m)}^c X_m^a) + (X_m^a X_{(n,m)}^c + X_{(n,m)}^c X_m^a)X_n^b \right) \tag{42}$$

Again we note that in the last formula the indices i and j are not dummy variables ,however the index (m,n) is either equal to m or to n , in any case the tensor which is contracted with ϵ^{abc} is symmetric with respect to the interchange of two of the indices (a,c) or (b,c) , hence the right hand side identically vanishes:

$$[C_m, C_n] = 0$$

It should be noted from (41) that all the Casimir operators commute with the generators of the largest copy. i.e:

$$[C_m, X_N^a] = 0 \tag{43}$$

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The family of commuting operators is the following set:

$$I = \{C_2, C_3, ...C_N, Z_N\} \tag{44}$$

Where the explicit expression for C_m the same as in (22). **Remark** :None of the formulas and results in this section depends on which representation of $su(2)$ sits on different sites . They are also independent of the particular algebra which we use . In fact what we have found is true for any irreducibe representation of any simple Lie algebra, provided that one adds to the set I all the higher order Casimir operators of the algebra.

5   Discussion

We have provided a mathematical basis in which the integrability and solvability of the models of [2] and their generalizations is explained. A very interesting problem is to find a mathematical formalism in which the integrability of systems like (1) whose Hamiltonians are not factorized can be explained. However the systems studied in [2] are perhaps a good testing ground and starting point for studying integrability in systems with local interactions. This may reminds us of the Mean Field Method which we use when we first encounter a new statistical system. It may be appropriate to call such systems Mean Field Integrable Models.

6   Acknowledgments

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